Characterization of Word-Representable Near-Triangulations

Suchanda Roy and Ramesh Hariharasubramanian
r.suchanda@iitg.ac.in, ramesh_h@iitg.ac.in
Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam 781039, India

Abstract

A pair of letters x and y are said to *alternate* in a word w if, after removing all letters except for the copies of x and y from w, the resulting word is of the form $xyxy\dots$ (of even or odd length) or $yxyx\dots$ (of even or odd length). A graph $G = (V(G), E(G))$ is *word-representable* if there exists a word w over the alphabet $V(G)$ such that two distinct vertices $x, y \in V(G)$ are adjacent in G (i.e., $xy \in E(G)$) if and only if the letters x and y alternate in w. An equivalent characterization states that a graph is word-representable if and only if it admits a *semi-transitive orientation*, namely an acyclic orientation in which every directed path $v_1 \to v_2 \to \cdots \to v_k$ with $k \geq 3$ satisfies the condition that either v_1 and v_k are non-adjacent, or all arcs $v_i \to v_j$ exist for $1 \leq i < j \leq k$.

The notion of word-representable graphs, first introduced by Sergey Kitaev and Steven Seif in the setting of Perkins semigroups [11], has become a rich area of research with strong connections to algebra, graph theory, and combinatorics on words. This graph class, which generalizes important classes such as circle graphs, comparability graphs, and 3-colorable graphs, has been extensively studied in monographs [9, 10]. Computational tools like Glen's software have further supported their analysis and construction. Notably, recognizing whether a graph is word-representable or not is an NP-complete problem.

Word-representability of planar graphs remains one of the most challenging open problems in the area of word-representable graphs. Since all triangle-free planar graphs are known to be word-representable (see Theorems 1,3), recent research has primarily focused on planar graphs rich in triangles. In particular, several subclasses have been investigated, including polyomino triangulations [1], triangulations of rectangular polyominoes with a single domino [6], K_4-free near-triangulations [5], face subdivisions of triangular grid graphs [3], triangulations of grid-covered cylinder graphs [4] and chordal near-triangulations [8].

In this work, we present a complete characterization of word-representable near-triangulations in terms of forbidden induced subgraphs. Recall that a *near-triangulation* is a planar graph in which every face is a triangle except the outer face. Our result not only unifies and generalizes all previously known characterizations for the subclasses mentioned above, but also corrects two inaccuracies in earlier works [5, 12]. This provides a comprehensive understanding of word-representability within the broader class of near-triangulations.

Keywords: semi-transitive orientation, near-triangulation, minimal forbidden subgraph, word-representable graph.

Characterization of Word-Representable Near-Triangulations

In this work, we establish a complete characterization of word-representable near-triangulations. The motivation of our result arises from three classical results stated below.

Theorem 1 (Grötzsch)**.** *[14] Every triangle-free planar graph is* 3*-colorable.*

Theorem 2. *[2] Every planar graph with at most* 3*-triangles is* 3*-colorable.*

Theorem 3. *[7] Every* 3*-colorable graph is word-representable.*

Together, these results show that planar graphs with few triangles pose no obstacle to word-representability. The challenge therefore, lies in understanding planar graphs rich in triangles—most notably, near-triangulations.

In the context of near-triangulations [13], Salam et al. introduced a decomposition of near-triangulations into W-components (i.e, 2-connected components free of both edge separators and triangle separators).

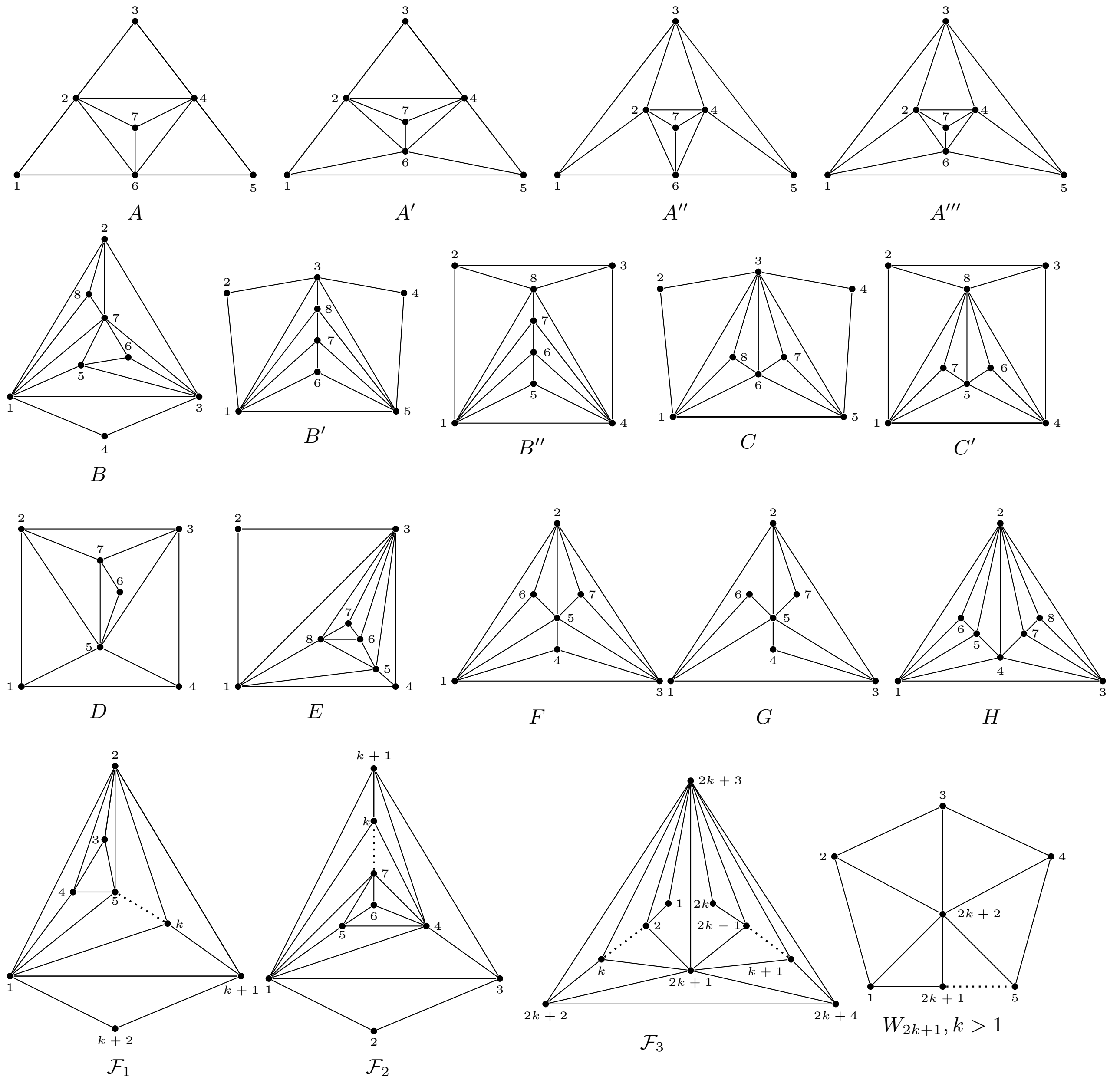

Figure 1: Forbidden induced subgraphs for word-representable near-triangulations; the last column contains infinite families.

However, from the viewpoint of word-representability, this decomposition has a limitation: the class of word-representable graphs is not closed under gluing two word-representable graphs along a clique. Consequently, a structural characterization of word-representable near-triangulations is equivalent to identifying the induced subgraphs whose exclusion guarantees the closure property of word-representable W-components of a near-triangulation under clique glueing.

As part of this work, we also clarify two inaccuracies in earlier papers [5, 12]. In particular, we disprove the claim from [5] that *a K_4-free near-triangulation is word-representable if and only if it is perfect.* We give an explicit counterexample (Figure 2) which is K_4-free, non-perfect but still 3-colorable as well as word-representable. Also, we provide the correct characterization for K_4-free word-representable near-triangulations as follows.

Theorem 4. *A K_4-free near-triangulation is word-representable if and only if it is odd-wheel-free i.e., contains no induced odd wheel W_{2k+1} (see Figure 1).*

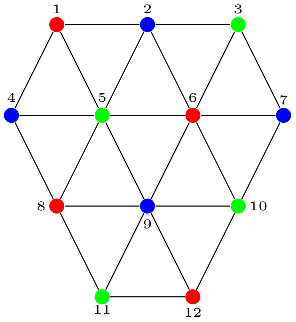

Figure 2: An example of a K_4-free 3-colorable as well as word-representable non-perfect graph. The non-perfectness becomes evident after deleting vertices 5, 6, and 7, which leaves an odd hole.

Additionally, we identify a new minimal non-word-representable graph on seven vertices (A'' in Figure 1), which does not appear among the all 25 minimal non-word-representable graphs on seven vertices listed by Kitaev et al. in [12].

These observations lead to our main result given below:

Theorem 5. *Let G be a near-triangulation. Then G is word-representable if and only if it contains none of the graphs from Figure 1 as an induced subgraph.*

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